TRINITY COLLEGE

Year 12 Mathematics Specialist
Test 4 2019

Section 1 Calculator Free

Integration

STUDENT'S NAME Solutions

DATE: Monday 1 July **TIME:** 33 minutes **MARKS**: 33

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, formula page

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

Determine

(a) $\int \frac{3x-2}{3x^2-4x+5}\,dx$ [2]

$f(x) = 3x^2 - 4x + 5$

$f'(x) = 6x - 4$

$= \frac{1}{2} \int \frac{2(3x-2)}{3x^2-4x+5}\,dx$

$= \frac{1}{2} \ln|3x^2-4x+5| + C$

(b) $\int \cos 2x \sin^3 2x\,dx$ [3]

$= \frac{1}{2} \int 2\cos 2x\,[\sin 2x]^3\,dx$

$= \frac{1}{2} \frac{[\sin 2x]^4}{4} + C$

$= \frac{1}{8} \sin^4 2x + C$

2. (6 marks)

(a) Express $\dfrac{5x-11}{(x+2)(2x-3)}$ in the form $\dfrac{a}{x+2}+\dfrac{b}{2x-3}$ [3]

$$\Rightarrow \quad 5x-11 = a(2x-3) + b(x+2) \quad (1)$$

$$\Rightarrow \quad 5x-11 = (2a+b)x + (-3a+2b)$$

$$\Rightarrow \quad 2a+b = 5$$
$$-3a+2b = -11$$

$$\Rightarrow \quad 4a+2b = 10$$
$$-3a+2b = -11$$

$$\Rightarrow \quad 7a = 21$$
$$a = 3$$
$$b = -1$$

Alternative method from (1)

$x = -2 \Rightarrow -21 = a(-7)$

$a = 3$

$x = \frac{3}{2} \Rightarrow -\frac{7}{2} = b\left(\frac{7}{2}\right)$

$b = -1$

$$\therefore \frac{5x-11}{(x+2)(2x-3)} = \frac{3}{x+2} - \frac{1}{2x-3}$$

(b) Hence, determine $\displaystyle\int \frac{5x-11}{(x+2)(2x-3)}\,dx$ [3]

$$= \int \frac{3}{x+2} - \frac{1}{2x-3} \, dx$$

$$= 3\ln|x+2| - \frac{1}{2}\ln|2x-3| + C$$

3. (9 marks)

Determine

(a) $\int \frac{x^2}{x-1}\,dx$ degree is higher on top [4]

∴ improper fraction

$$= \int x+1 + \frac{1}{x-1}\,dx$$

$$\begin{array}{r|l} & x + 1 \\ \hline x-1 & x^2 + 0x + 0 \\ & \underline{x^2 - x} \\ & x + 0 \\ & \underline{x - 1} \\ & 1 \end{array}$$

$$= \frac{x^2}{2} + x + \ln|x-1| + C$$

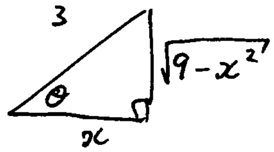
(b) $\int \sqrt{9-x^2}\, dx$ let $x = 3\cos\sigma$ [5]

$$\frac{dx}{d\theta} = -3\sin\theta$$

$$= \int \sqrt{9-(3\cos\theta)^2} \times (-3\sin\theta)\, d\theta$$

$$= \int 3\sin\theta \times (-3\sin\theta)\, d\theta$$

$$= -9 \int \sin^2\theta \; d\theta$$



$$= -\frac{9}{2} \int 1 - \cos 2\theta \; d\theta$$

$$\cos\theta = \frac{x}{3}$$

$$\sin\theta = \frac{\sqrt{9-x^2}}{3}$$

$$= \frac{-9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{-9}{2}\left[\cos^{-1}\left(\frac{x}{3}\right) - \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot \frac{\sqrt{9-x^2}}{3} \cdot \frac{x}{3}\right] + c$$

$$= -\frac{9}{2}\cos^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + c$$

4. (7 marks)

(a) By using an appropriate trigonometric substitution, simplify in terms of u, the expression $4-x^2$ where $x = 2\sin u$ [2]

$$4-(2\sin u)^2$$

$$= 4-4\sin^2 u$$

$$= 4(1-\sin^2 u)$$

$$= 4\cos^2 u$$

(b) Hence, evaluate $\int_1^{\sqrt{3}} \frac{x}{4-x^2}\,dx$ exactly [5]

let $x = 2\sin u$

$\frac{dx}{du} = 2\cos u$

$x = \sqrt{3}$ $\quad \frac{\sqrt{3}}{2} = \sin u$

$\Rightarrow u = \pi/3$

$x = 1$ $\quad \frac{1}{2} = \sin u$

$\Rightarrow u = \pi/6$

$$= \int_{\pi/6}^{\pi/3} \frac{2\sin u}{4\cos^2 u} \cdot 2\cos u \, du$$

$$= -\int_{\pi/6}^{\pi/3} \frac{-\sin u}{\cos u} \, du$$

$$= -\Big[\ln|\cos u|\Big]_{\pi/6}^{\pi/3}$$

$$= -\left[\ln \cos \pi/3 - \ln \cos \pi/6\right]$$

$$= -\left[\ln \tfrac{1}{2} - \ln \tfrac{\sqrt{3}}{2}\right]$$

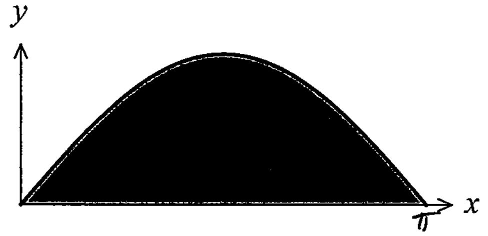
$$= -\ln \frac{1}{\sqrt{3}}$$

$$= \ln \sqrt{3}$$

$$= \frac{1}{2}\ln 3$$

5. (6 marks)

Consider the area enclose by $\sin x$ and the x axis shown below.



(a) Determine the exact volume when the shaded area is rotated about the x axis. [3]

$$V_x = \pi \int_0^{\pi} [\sin x]^2 \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} 1 - \cos 2x \, dx$$

$$= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{\pi}{2} \left[\left(\pi - \underbrace{\frac{\sin 2\pi}{2}}_{=0} \right) - \left(0 - \cancel{\frac{\sin 0}{2}} \right) \right]$$

$$= \frac{\pi^2}{2} \text{ units}^3$$

(b) Determine the exact volume when the shaded area is rotated about the y axis. [3]

You may find the following formulas useful:

$$\frac{d}{dx}(\sin x - x\cos x) = x \sin x$$

$$Vol_y = 2\pi \int_c^d x[f(x)] \, dx$$

$$V_y = 2\pi \int_0^{\pi} x . \sin x \, dx$$

$$= 2\pi \left[\sin x - x \cos x \right]_0^{\pi}$$

$$= 2\pi \left[\left(\underbrace{\sin \pi}_{=0} - \pi \cos \pi \right) - \left(\underbrace{\cancel{\sin 0 - 0 \cos 0}}_{=0} \right) \right]$$

$$= 2\pi^2 \text{ units}^3$$

TRINITY COLLEGE

Year 12 Mathematics Specialist
Test 4 2019

Section 2 Calculator Assumed
Integration

STUDENT'S NAME ______________________________

DATE: Monday 1 July **TIME:** 17 ~~20~~ minutes **MARKS:** 17

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, formula page

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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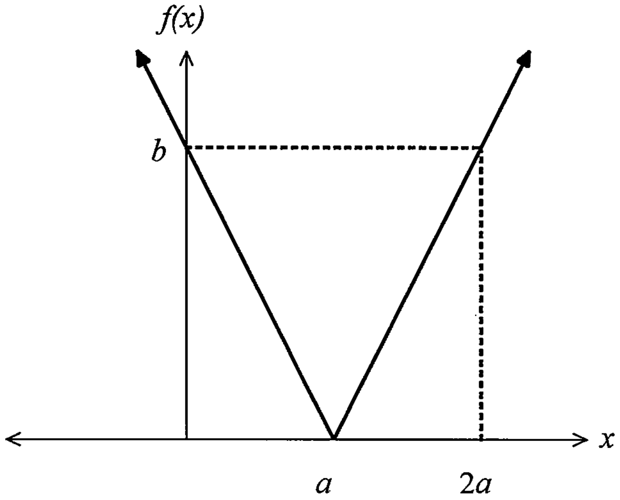
6. (8 marks)

Function f is defined by its graph shown below. The constants $a, b > 0$ where $b > a$.



(a) Determine the defining rule for function $f(x)$ in terms of a, b . [3]

$$f(x) = \left| \frac{b}{a}(x-a) \right|$$

or

$$f(x) = \frac{b}{a} \left| x - a \right|$$

(b) By using the substitution $u = 2x - a$, determine an expression, in terms of a, b, for the value of $\int_{\frac{a}{2}}^{a} f(2x-a)dx$ [5]

$u = 2x - a$

$\frac{du}{dx} = 2$

$$= \int_0^a -\frac{b}{a}(u-a) \cdot \frac{1}{2}\, du$$

$x = a \quad u = a$

$x = \frac{a}{2} \quad u = 0$

$$= \int_0^a -\frac{b}{2a}u + \frac{b}{2} \quad du$$

$$= \left[-\frac{b}{4a}u^2 + \frac{b}{2}u\right]_0^a$$

$$= \left(-\frac{b}{4a}a^2 + \frac{b}{2}a\right) - 0$$

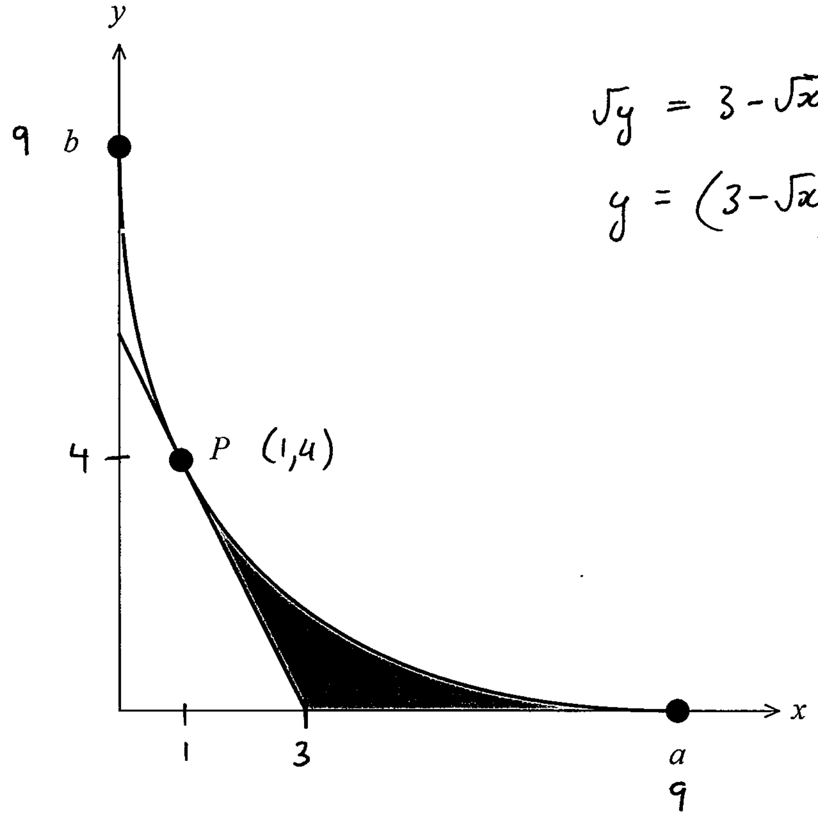
$$= -\frac{ba}{4} + \frac{ba}{2}$$

$$= \frac{-ba + 2ba}{4}$$

$$= \frac{ba}{4}$$

7. (9 marks)

The diagram shows the curve with equation $\sqrt{x}+\sqrt{y}=3$ where points a, b are the intercepts of this curve. A tangent is drawn to the curve at point $P\,(1,4)$ with equation $2x+y=6$.



$$\sqrt{y} = 3-\sqrt{x}$$
$$y = (3-\sqrt{x})^2$$

The shaded area on the diagram is bounded by the curve, the tangent and the x axis.

(a) Determine the exact area of the shaded region. [5]

A = total − triangle

$$= \int_1^9 (3-\sqrt{x})^2\,dx - \frac{1}{2}\times 2\times 4$$

$$= 4 \text{ units}^2$$

Or, you can use horizontal rectangles

$$A = \int_0^4 (3-\sqrt{y})^2 - \left(\frac{6-y}{2}\right)\,dy$$

$$= 4 \text{ units}^2$$

The shaded region is then rotated about the x axis.

(b) Calculate the volume of the resulting solid, correct to 0.01 cubic units. [4]

$$V = V_{tot} - V_{cone}$$

$$= \pi\int_1^9 \left[(3-\sqrt{x})^2\right]^2 dx - \pi\int_1^3 \left[6-2x\right]^2 dx$$

$$= 20.11 \text{ units}^3$$

Or, you can use a shell method

$$V = 2\pi\int_0^4 y \cdot \left[(3-\sqrt{y})^2 - \left(\frac{6-y}{2}\right)\right] dy$$

$$= 20.11 \text{ units}^3$$